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Error Bounds for Gauss Type Quadrature Formulae Related to Spaces of Splines with Equidistant Knots

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Error bounds for the Gauss type quadrature formulae Q_n^G, Q_{n+1}^L and Q_{n+1}^R (Gauss, Lobatto and Radau formulae) related to the spaces of polynomial spline functions of degree $r-1$ with equidistant knots are obtained. It is shown that these quadrature rules are asymptotically optimal in the Sobolev space W_∞^r for all r, and in W_p^r ($1 \leq p \leq \infty$) for odd r. Some inequalities involving the Gaussian nodes and weights are also established.

1. INTRODUCTION AND RESULTS

The object of this paper is to study quadrature formulae of the type

$$Q[f] := \sum_{i=1}^{n} a_i f(\xi_i) \qquad (0 \leq \xi_1 < \cdots < \xi_n \leq 1), \tag{1.1}$$

which serve as estimates for the definite integral

$$I[f] := \int_0^1 f(x)\, dx.$$

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We consider quadrature formulae which are exact for polynomial splines of degree $r-1$ with equidistant knots i/N $(i=1, ..., N-1)$, and which use a minimal number of function values $f(\xi_i)$. These formulae are called Gauss type formulae, related to this space of spline functions.

One of the fundamental questions in the theory of quadrature formulae is, for a given normed linear space X of functions defined on $[0, 1]$, to minimize the remainder $R := I - Q$ with respect to the parameters $\{a_i\}_{i=1}^n$ and $\{\xi_i\}_{i=1}^n$, i.e., to find

$$\mathscr{E}_n(X) := \inf_Q \sup_{\|f\|_X \leqslant 1} |R[f]|.$$

If the quantity $\mathscr{E}_n(X)$ is attained for a quadrature formula Q^{opt} of the type (1.1), Q^{opt} is said to be the optimal quadrature formula of the type (1.1) for the space X. Let

$$\tilde{W}_p^r := \{f \in C^{r-1}(\mathbb{R}): f \text{ 1-periodic}, f^{(r-1)} \text{ abs. cont.}, \|f^{(r)}\|_p < \infty\},$$

$$W_p^r := \{f \in C^{r-1}[0, 1]: f^{(r-1)} \text{ abs. cont.}, \|f^{(r)}\|_p < \infty\},$$

where

$$\|f\|_p := \left(\int_0^1 |f(u)|^p \, du\right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty, \text{ and } \|f\|_\infty := \sup_{u \in (0, 1)} \text{vrai } |f(u)|.$$

The problems of existence and uniqueness of optimal quadrature formulae for the periodic Sobolev spaces $X = \tilde{W}_p^r$ have been solved by Motornyĭ [11] for $p = \infty$ and $p = 1$, r odd, by Ligun [10] for $p = 1$, r even, and in the general case by Žensykbaev [16]. In all these cases the optimal quadrature formula is the rectangle rule. The exact error constants $\mathscr{E}_n(X)$ in this case are (see Žensykbaev [16], p. 1070)

$$\mathscr{E}_n(\tilde{W}_p^r) = \frac{1}{n^r} \inf_c \|B_r - c\|_q \qquad \text{for} \quad r \geqslant 1 \quad \text{and} \quad 1 \leqslant p \leqslant \infty. \tag{1.2}$$

Special cases are

$$\mathscr{E}_n(\tilde{W}_p^r) = \frac{1}{n^r} \|B_r\|_q \qquad \text{for odd } r \quad \text{and} \quad 1 \leqslant p \leqslant \infty, \tag{1.3}$$

$$\mathscr{E}_n(\tilde{W}_\infty^r) = \frac{K_r}{(2\pi n)^r} \qquad \text{for} \quad r \geqslant 1, \tag{1.4}$$

where B_r are the Bernoulli polynomials, $1/p + 1/q$, and

$$K_r = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m(r+1)}}{(2m+1)^{r+1}} \tag{1.5}$$

are Favard's constants. For the non-periodic case $X = W_p^r$, the problems of existence and uniqueness are also solved (even in a more general setting, for quadrature formulae that involve derivatives and boundary terms) by Bojanov [1], [2] (see also Žensykbaev [17]); however, the optimal quadrature formulae and the corresponding error bounds are not known, in general. Obviously, $\mathscr{E}_n(W_p^r) \geq \mathscr{E}_n(\tilde{W}_p^r)$, and it is known that

$$\lim_{n \to \infty} \frac{\mathscr{E}_n(\tilde{W}_p^r)}{\mathscr{E}_n(W_p^r)} = 1 \qquad \text{for} \quad 1 < p \leq \infty$$

(this follows from Brass [4]; for $p = 1$, we could not find a reference concerning this question).

The purpose of this paper is to show that, for the spaces W_p^r with $p = \infty$ or with odd r, the Gauss type quadrature rules related to spaces of spline functions of degree $r-1$ with equidistant knots have the same asymptotic behaviour as the optimal quadrature rules. In particular, this proves a conjecture proposed by one of the authors in [12] about asymptotic optimality of the Gaussian quadrature rules in W_∞^r. To prove this, we determine upper bounds for the errors of the Gauss type rules for all classes W_p^r. These error bounds are also upper bounds for the best error constants in these spaces (i.e., for $\mathscr{E}_n(W_p^r)$); for other bounds for $\mathscr{E}_n(W_p^r)$, $p = 2, \infty$, see Strauss [13]. We also establish some inequalities involving the Gaussian nodes and weights.

For $r, N \in \mathbb{N}$, define

$$S_{r-1,N} := \{f \in C^{r-2}[0,1] : f_{|((i-1)/N,\, i/N)} \in \pi_{r-1},\ i = 1, \ldots, N\}, \tag{1.6}$$

where π_{r-1} denotes the set of all algebraic polynomials of degree strictly less than r. The dimension of $S_{r-1,N}$ is $N + r - 1$. For $N = 2n + 1 - r$ (i.e., $n = (\dim S_{r-1,N})/2$, if $\dim S_{r-1,N}$ is even) there exist unique quadrature formulae of the type

$$Q_n^G[f] := \sum_{i=1}^{n} a_i^G f(\xi_i^G) \qquad (0 < \xi_1^G < \cdots < \xi_n^G < 1), \tag{1.7}$$

$$Q_{n+1}^L[f] := \sum_{i=0}^{n} a_i^L f(\xi_i^L) \qquad (0 = \xi_0^L < \cdots < \xi_n^L = 1), \tag{1.8}$$

which are exact for every function from the space (1.6), called the Gauss and Lobatto quadrature formulae, related to this space. For $N = 2n + 2 - r$

(i.e., $n=(\dim S_{r-1,N}-1)/2$, if $\dim S_{r-1,N}$ is odd) there exist unique quadrature formulae of the type

$$Q_{n+1}^{R0}[f]:=\sum_{i=0}^{n} a_i^{R0} f(\xi_i^{R0}) \qquad (0=\xi_0^{R0}<\cdots<\xi_n^{R0}<1), \tag{1.9}$$

$$Q_{n+1}^{R1}[f]:=\sum_{i=1}^{n+1} a_i^{R1} f(\xi_i^{R1}) \qquad (0<\xi_1^{R1}<\cdots<\xi_{n+1}^{R1}=1), \tag{1.10}$$

which are exact for every function from the space (1.6), and are referred to as the left and right Radau formulae, related to this space. The Gauss, Lobatto and Radau formulae are also called Gauss type quadrature formulae. (For the existence and unicity of these formulae, see, e.g., [8].)

We state below the main results of this paper. Let

$$\mathscr{E}(Q, W_p^r):=\sup_{\|f^{(r)}\|_p\leqslant 1} |I[f]-Q[f]|,$$

$$\bar{B}_r(x):=\begin{cases} B_r(x) & \text{for odd } r \\ B_r(x)-B_r(1/4) & \text{for even } r, \end{cases}$$

and

$$c_{r,q}:=\int_0^1 |\bar{B}_r(t)|^q \operatorname{sign} \bar{B}_r(t)\, dt \Big/ \int_0^1 |\bar{B}_r(t)|^q\, dt$$

for $1\leqslant q<\infty$ ($c_{r,q}=0$ for arbitrary q and odd r). The following theorems hold for $1\leqslant p\leqslant\infty$ ($1/p+1/q=1$) and $r\geqslant 1$.

THEOREM 1.1. *For the error of the Gauss quadrature formula* (1.7) *related to* $S_{r-1,N}$ *with* $N=2n+1-r$, *there holds*

$$\mathscr{E}(Q_n^G, W_p^r)\leqslant\frac{1}{(n-(r-1)/2)^r}\|\bar{B}_r\|_q\left(1+\frac{c_{r,q}}{2n+1-r}\right)^{1/q}$$

with equality for $r=1$ *only.*

THEOREM 1.2. *For the error of the Lobatto quadrature formula* (1.8) *related to* $S_{r-1,N}$ *with* $N=2n+1-r$, *there holds*

$$\mathscr{E}(Q_{n+1}^L, W_p^r)\leqslant\frac{1}{(n-(r-1)/2)^r}\|\bar{B}_r\|_q\left(1-\frac{c_{r,q}}{2n+1-r}\right)^{1/q}$$

with equality for $r=1,2$ *only.*

THEOREM 1.3. *For the error of the Radau quadrature formulae* (1.9) *and* (1.10) *related to* $S_{r-1,N}$ *with* $N=2n+2-r$, *there holds* (*with* $i=0,1$)

$$\mathscr{E}(Q_{n+1}^{R_i}, W_p^r) \leqslant \frac{1}{(n-(r/2)+1)^r} \|\bar{B}_r\|_q$$

with equality for $r=1$ *only*.

For $q=\infty$, the terms $(\cdots)^{1/q}$ have to be replaced by 1. Some special cases of the constants involved above are the following ones: For arbitrary r, there holds $\|\bar{B}_r\|_1 = K_r/(2\pi)^r$, with K_r as in (1.5), and for even r, there holds $\|\bar{B}_r\|_1 = 4\,|B_{r+1}(1/4)|$ and $c_{r,1} = (-1)^{r/2-1}\,|B_r(1/4)|/\|\bar{B}_r\|_1$; esp., $c_{2,1}=1/3$ and $c_{4,1}=-7/75$.

As an immediate consequence of (1.2)–(1.4) and the above theorems, we obtain the following result about the asymptotic optimality of the Gauss type formulae.

THEOREM 1.4. *The Gauss type quadrature formulae related to* $S_{r-1,N}$ *are asymptotically optimal in* W_p^r *for* $r \geqslant 1$ *if* $p=\infty$, *and for all* p *with* $1 \leqslant p \leqslant \infty$ *if* r *is odd. More precisely, for these values of* r *and* p, *there holds*

$$\lim_{n\to\infty} \frac{\mathscr{E}(Q_n^*, W_p^r)}{\mathscr{E}_n(W_p^r)} = 1$$

with * *standing for* G, L, $R0$ *and* $R1$.

For other cases where the Gauss type formulae are asymptotically optimal, see Section 4. Up to now, we have considered the error only, but it is also possible to give estimates for the weights of these quadrature formulae.

THEOREM 1.5. *With* * *standing for* G, L *and* $R0$, *and with the corresponding* N, *the Gaussian weights* (a_i^*) *and the Gaussian nodes* (ξ_i^*) *satisfy the following inequalities*:

(a) *for odd* $r \geqslant 3$,

$$\sum_{0 \leqslant \xi_\nu^* < k/N} a_\nu^* > \frac{k-1}{N} \quad \textit{for} \quad k=1, \ldots, N, \tag{1.11}$$

$$\sum_{0 \leqslant \xi_\nu^* \leqslant l/N} a_\nu^* < \frac{l+1}{N} \quad \textit{for} \quad l=0, \ldots, N-1; \tag{1.12}$$

(b) *for even* $r \geqslant 2$,

$$\sum_{0 \leqslant \xi_\nu^* < (2k-1)/(2N)} a_\nu^* \geqslant \frac{2k-3}{2N} \quad \textit{for} \quad k = 2, ..., N, \tag{1.13}$$

$$\sum_{0 \leqslant \xi_\nu^* \geqslant (2l-1)/(2N)} a_\nu^* \leqslant \frac{2l+1}{2N} \quad \textit{for} \quad l = 1, ..., N-1 \tag{1.14}$$

(*equality is possible only in the Lobatto case for* $r = 2$).

Similar inequalities hold for (a_i^{R1}) and (ξ_i^{R1}). Thec can be derived from (1.11)–(1.14) by taking into account the identity $Q_{n+1}^{R1}[f(\cdot)] \equiv Q_{n+1}^{R0}[f(1-\cdot)]$.

In Section 2 we give some definitions and known results, including the relationship between monosplines and quadrature formulae, the Budan-Fourier theorem for splines and some properties of the Bernoulli polynomials. Theorems 1.1–1.5 are proved in Section 3. In Section 4 some corollaries and concluding remarks are given.

2. Preliminaries

In the following, we assume that $r \geqslant 2$. For $r = 1$, with the convention that $s(x_k) := (s(x_k -) + s(x_k +))/2$ for $s \in S_{0,N}$ and $k = 1, ..., N-1$, the Gauss type formulae for splines with equidistant knots are the compound midpoint rule (Gauss), the compound trapezoidal rule (Lobatto) and a kind of truncated compound trapezoidal rules in the Radau case.

The basic ingredient of the proofs of our results for $r \geqslant 2$ is the extension of the classical Budan-Fourier Theorem for polynomials to splines. Before formulating this theorem, we recall some definitions, following de Boor and Schoenberg's paper [3].

For $\bar{a} := (a_1, ..., a_n) \in \mathbb{R}^n$, denote by $S^-(\bar{a})$ the number of sign changes in the sequence $a_1, a_2, ..., a_n$, ignorizing zeros in this sequence. In contrast, $S^+(\bar{a})$ denotes the maximal number of sign changes in this sequence, provided that the zeros are taken with appropriate signs. Analogously, if f is a real-valued function on some domain $G \subset \mathbb{R}$, then

$$S_G^-(f) := \sup\{S^-(f(t_1), f(t_2), ..., f(t_n)) \colon \{t_i\}_{i=1}^n \in G, n \in \mathbb{N}, t_1 < \cdots < t_n\}.$$

For a spline function f of degree n with simple knots, $Z_{f^{(n)}}(a, b) = S_{(a,b)}^-(f^{(n)})$ denotes the number of strong sign changes of $f^{(n)}$ in (a, b), and $Z_f(a, b)$ denotes the total number of zeros of f in (a, b), counting their multiplicities in the way defined in [3].

THEOREM 2.1 (Budan-Fourier Theorem for Splines). *If f is a polynomial spline of exact degree n on (a, b) (i.e., of degree n with $f^{(n)}(t) \neq 0$ for some $t \in (a, b)$) with finitely many (active) knots in (a, b), all simple, then*

$$Z_f(a, b) \leqslant Z_{f^{(n)}}(a, b) + S^-(f(a), f'(a), ..., f^{(n-1)}(a), f^{(n)}(\sigma+)) \\ - S^+(f(b), f'(b), ..., f^{(n-1)}(b), f^{(n)}(\tau-)),$$

where $[\sigma, \tau] \subset [a, b]$ is the largest interval such that $f^{(n)}(\sigma+) \neq 0$ and $f^{(n)}(\tau-) \neq 0$.

Remark. The above formulation of the Budan-Fourier theorem is a slight modification of that given by de Boor and Schoenberg in [3], which uses $a+$ instead of $\sigma+$ and $b-$ instead of $\tau-$. However, as simple examples show, this may give a wrong result if $\sigma > a$ or $\tau < b$. The reason is an inaccuracy in the second equality of (7), p. 8 in [3]. There, the right-hand side may not be zero, and one has to add correction terms, which yields

$$\begin{aligned} Z_f(a, b) \leqslant Z_{f^{(n)}}(a, b) &+ S^-(f(a), f'(a), ..., f^{(n-1)}(a), f^{(n)}(a+)) \\ &- S^+(f(b), f'(b), ..., f^{(n-1)}(b), f^{(n)}(b-)) \\ &+ S^-(f^{(j)}(a), f^{(n)}(\sigma+)) \\ &+ S^-(f^{(k)}(b), (-1)^{n-k} f^{(n)}(\tau-)), \end{aligned} \tag{2.1}$$

where $f^{(j)}(a)$ and $f^{(k)}(b)$ are the highest non-zero derivatives at a and b, respectively. (If all derivatives vanish at a, then replace $f^{(j)}(a)$ by 0, and analogously at b.) The corrected estimate (2.1) is equivalent to the estimate given in Theorem 2.1.

For a given set $\Delta = \{x_i\}_{i=1}^{N-1}$ $(0 < x_1 < \cdots < x_{N-1} < 1)$, we consider the space of spline functions of degree $r-1$ with simple knots belonging to Δ:

$$S_{r-1,\Delta} := \{f \in C^{r-2}[0, 1] : f_{|(x_{i-1}, x_i)} \in \pi_{r-1}, i = 1, ..., N\},$$

where $x_0 := 0$ and $x_N := 1$. $S_{r-1,\Delta}$ is a linear space of dimension $N + r - 1$ with a basis given by

$$\{1, x, ..., x^{r-1}, (x - x_1)_+^{r-1}, ..., (x - x_{N-1})_+^{r-1}\},$$

where $u_+ := \max\{u, 0\}$.

Next, we recall the connection between quadrature formulae and monosplines. The quadrature formula Q is said to have algebraic degree of precision m, if $I[f] = Q[f]$ for each $f \in \pi_m$, but $I[(\cdot)^{m+1}] \neq Q[(\cdot)^{m+1}]$. In the following, we always assume that Q has algebraic degree of precision at least $r-1$. For quadrature formulae which are exact for some spline space of degree $r-1$, this is trivially the case. According to Peano's theorem (see, e.g., Brass [5]),

$$R[f]=\int_0^1 K_r(t)f^{(r)}(t)\,dt \qquad \text{for} \quad f\in W_1^r, \tag{2.2}$$

where $K_r(t)=R[(\cdot-t)_+^{r-1}/(r-1)!]$ is the Peano kernel of order r. Clearly, for $t\in(0,1)$,

$$K_r(t)=\frac{(1-t)^r}{r!}-\frac{1}{(r-1)!}\sum_{k=1}^{n} a_k(\xi_k-t)_+^{r-1}, \tag{2.3}$$

or, using the identity $(x-t)_+^{r-1}=(-1)^r[(t-x)_+^{r-1}-(t-x)^{r-1}]$, equivalently,

$$K_r(t)=(-1)^r\left\{\frac{t^r}{r!}-\frac{1}{(r-1)!}\sum_{k=1}^{n} a_k(t-\xi_k)_+^{r-1}\right\}. \tag{2.4}$$

According to (2.2),

$$\mathscr{E}(Q,W_p^r)=\sup_{\|f^{(r)}\|_p\leqslant 1}|R[f]|=\|K_r\|_q, \qquad \text{where} \quad \frac{1}{p}+\frac{1}{q}=1. \tag{2.5}$$

The function K_r is a monospline of degree r with simple knots $\{\xi_i\colon \xi_i\in(0,1)\}$. Moreover, it is easily seen that

$$K_r^{(j)}(0)=0 \qquad \text{for} \quad j=0,\dots,r-1-\alpha \tag{2.6}$$

and

$$K_r^{(j)}(1)=0 \qquad \text{for} \quad j=0,\dots,r-1-\beta, \tag{2.7}$$

where

$$\alpha=\begin{cases}0 & \text{for } \xi_1>0\\ 1 & \text{for } \xi_1=0\end{cases}, \qquad \text{and} \qquad \beta=\begin{cases}0 & \text{for } \xi_n<1\\ 1 & \text{for } \xi_n=1\end{cases}.$$

Obviously, for fixed $\tau\in(0,1)$, since $K_r(\tau)=R[(\cdot-\tau)_+^{r+1}/(r-1)!]$ we have that Q is exact for the spline $f(x)=(x-\tau)_+^{r-1}$ if and only if $K_r(\tau)=0$. Therefore, in order that the quadrature formula (1.1) have maximal "spline degree of precision", i.e., that it is exact for a space $S_{r-1,\Delta}$ of the highest possible dimension, it is necessary and sufficient that the corresponding monospline K_r have a maximal number of simple zeros in $(0,1)$. As a consequence of the fundamental theorem of algebra for monosplines satisfying boundary conditions (Karlin and Micchelli [8], Theorem 0.1), every monospline of the type (2.3)–(2.4) has at most $2n-r-\alpha-\beta$ simple zeros in $(0, 1)$, and, conversely, given any $2n-r-\alpha-\beta$ distinct points in $(0, 1)$, say $\tau_1 < \tau_2 < \cdots < \tau_{2n-r-\alpha-\beta}$, there exists a unique monospline K_r of the form (2.4), that vanishes at these points and satisfies (2.6)–(2.7). The corresponding quadrature formula Q is of the type (1.1), and it is exact for the space $S_{r-1,\Delta}$ with $\Delta = \{\tau_i\}_{i=1}^{2n-r-\alpha-\beta}$. Clearly, $S_{r-1,\Delta}$ has dimension $2n-\alpha-\beta$, but Q uses only n function values, i.e., Q is a double precision quadrature formula. If $\xi_1 > 0$ and $\xi_n < 1$, then this formula is refered to as the Gauss formula related to the space $S_{r-1,\Delta}$; if $\xi_1 = 0$ and $\xi_n = 1$, it is called the Lobatto formula related to $S_{r-1,\Delta}$, and in the cases $\xi_1 = 0$ and $\xi_n < 1$, or $\xi_1 > 0$ and $\xi_n = 1$, we have the left and the right Radau formulae. The weights of the Gauss, Lobatto and Radau formulae are always positive (see [8], Theorem 7.1).

Denote the corresponding r-th Peano kernels by $K^G_{n,r}$, $K^L_{n,r}$, $K^{R0}_{n,r}$ and $K^{R1}_{n,r}$, respectively. According to (2.5), to find error bounds, one has to estimate the L_q-norms of these monosplines. We do this by a pointwise comparison with some modified Bernoulli monosplines.

The Bernoulli polynomials B_ν are defined recursively by

$$B_0(x) := 1, \quad B'_\nu(x) = B_{\nu-1}(x), \quad \text{and} \quad \int_0^1 B_\nu(u)\,du = 0 \quad \text{for} \quad \nu \geqslant 1.$$

In the next lemma we list some properties of the Bernoulli polynomials, which will be used repeatedly in the sequel.

LEMMA 2.1.

(i) $B_\nu(x) = \sum_{k=0}^{\nu} \frac{B_{\nu-k}(0)}{k!} x^k \qquad (\nu \geqslant 0)$,

(ii) $\operatorname{sign} B_{2\nu}(0) = \operatorname{sign} B_{2\nu}(1) = (-1)^{\nu-1} \qquad (\nu \geqslant 1)$,

(iii) $B_{2\nu+1}(0) = B_{2\nu+1}(\tfrac{1}{2}) = B_{2\nu+1}(1) = 0 \qquad (\nu \geqslant 1)$,

(iv) $B_1(0) = -B_1(1) = -\tfrac{1}{2}, \qquad B_1(\tfrac{1}{4}) = -B_1(\tfrac{3}{4}) = -\tfrac{1}{4}, \qquad B_1(\tfrac{1}{2}) = 0$,

(v) $\operatorname{sign} B_{2\nu}(\tfrac{1}{4}) = \operatorname{sign} B_{2\nu}(\tfrac{1}{2}) = \operatorname{sign} B_{2\nu}(\tfrac{3}{4}) = (-1)^\nu \qquad (\nu \geqslant 1)$,

(vi) $\operatorname{sign} B_{2\nu+1}(\tfrac{1}{4}) = -\operatorname{sign} B_{2\nu+1}(\tfrac{3}{4}) = (-1)^{\nu+1} \qquad (\nu \geqslant 0)$.

Note also that $x = 1/2$ is the unique zero of $B_{2\nu+1}$ in $(0, 1)$, while $B_{2\nu}$ has two simple zeros in $(0, 1)$, located symmetrically with respect to the middle of the interval. Denote by B^*_ν the 1-periodic extension of $B_{\nu|[0,1)}$. Then B^*_ν is a monospline of degree ν with leading term $x^\nu/\nu!$ and with simple knots at $0, \pm 1, \pm 2, \ldots$ (see (i)–(iv) and the definition of B_ν).

3. The Proofs

To prove Theorems 1.1–1.3, we compare the Peano kernels, $K^G_{n,r}$, $K^L_{n+1,r}$, $K^{R0}_{n+1,r}$ and $K^{R1}_{n+1,r}$ with properly chosen modified Bernoulli monosplines K, $\tilde{K}$, L, and $\tilde{L}$, which are defined by

$$K(x) = -\left(\frac{2}{N}\right)^r B_r^*\left(\frac{N}{2}x\right), \tag{3.1}$$

$$\tilde{K}(x) = -\left(\frac{2}{N}\right)^r B_r^*\left(\frac{N}{2}x + \frac{1}{2}\right), \tag{3.2}$$

$$L(x) = \left(\frac{2}{N}\right)^r \left(B_r^*\left(\frac{N}{2}x + \frac{1}{4}\right) - B_r\left(\frac{1}{4}\right)\right), \tag{3.3}$$

$$\tilde{L}(x) = \left(\frac{2}{N}\right)^r \left(B_r^*\left(\frac{N}{2}x + \frac{3}{4}\right) - B_r\left(\frac{3}{4}\right)\right). \tag{3.4}$$

We obtain pointwise estimates for the Peano kernels of the Gaussian quadrature formulae by these modified Bernoulli monosplines, and the pointwise estimates immediately imply norm estimates. For the convenience of the reader, and since they may be of independent interest, we state them in a corollary below. Their proof is a major part of the proofs of Theorems 1.1–1.3. (They are comparison theorems for monosplines, related in a sense to Karlin's global improvement theorem, see Karlin [7], and to comparison theorems for monosplines of Strauss [14], [15]. The comparison theorems of Strauss can be used to prove some special cases of the results obtained here, see Theorem 4.1 of [14]. But in contrast to these results, in our case the monosplines on the right-hand sides of the estimates do not all have maximal number of zeros.)

Corollary 3.1. *Let K and $\tilde{K}$ be defined as in* (3.1)–(3.2), *and L and $\tilde{L}$ as in* (3.3)–(3.4). *Then, for $x \in [0, 1]$, the following estimates hold.*

(a) *Let $N = 2n - r + 1$. The Peano kernels of the Gauss formulae satisfy*

$$|K^G_{n,r}(x)| \leqslant |K(x)| \quad \text{if } r = 4m - 1, \qquad |K^G_{n,r}(x)| \leqslant |\tilde{K}(x)| \quad \text{if } r = 4m + 1,$$

$$|K^G_{n,r}(x)| \leqslant |L(x)| \quad \text{if } r = 4m, \qquad |K^G_{n,r}(x)| \leqslant |\tilde{L}(x)| \quad \text{if } r = 4m + 2.$$

(b) *Let $N=2n-r+1$. The Peano kernels of the Lobatto formulae satisfy*

$$\begin{aligned}
|K^L_{n+1,r}(x)| &\leqslant |K(x)| && \textit{if}\quad r=4m+1,\\
|K^L_{n+1,r}(x)| &\leqslant |\tilde{K}(x)| && \textit{if}\quad r=4m-1,\\
|K^L_{n+1,r}(x)| &\leqslant |L(x)| && \textit{if}\quad r=4m+2,\\
|K^L_{n+1,r}(x)| &\leqslant |\tilde{L}(x)| && \textit{if}\quad r=4m.
\end{aligned}$$

(c) *Let $N=2n-r+2$. The Peano kernels of the left Radau formulae satisfy*

$$\begin{aligned}
|K^{R0}_{n+1,r}(x)| &\leqslant |K(x)| && \textit{if}\quad r=4m+1,\\
|K^{R0}_{n+1,r}(x)| &\leqslant |\tilde{K}(x)| && \textit{if}\quad r=4m-1,\\
|K^{R0}_{n+1,r}(x)| &\leqslant |L(x)| && \textit{if}\quad r=4m,\\
|K^{R0}_{n+1,r}(x)| &\leqslant |\tilde{L}(x)| && \textit{if}\quad r=4m+2.
\end{aligned}$$

The proofs of Theorems 1.1–1.3 are not essentially different from each other. For this reason, we give a detailed proof of Theorem 1.1 only, prove Theorem 1.2 by outlining the differences to the proof of Theorem 1.1, and sketch the proof of Theorem 1.3.

Proof of Theorem 1.1. The Peano kernel $K^G_{n,r}$ is a monospline of degree r with n simple knots $\{\xi^G_i\}^n_{i=1}$ in $(0, 1)$ and $N-1$ simple zeros

$$\tau_i=\frac{i}{N}, \qquad i=1, ..., N-1.$$

Moreover, $K^G_{n,r}$ satisfies

$$(K^G_{n,r})^{(j)}(0)=(K^G_{n,r})^{(j)}(1)=0, \qquad j=0, ..., r-1. \tag{3.5}$$

We consider the cases of odd and even r separately.

Case A: $r=2s+1$. In this case $N=2n-r+1=2(n-s)$. Let K and $\tilde{K}$ be defined as in (3.1) and (3.2). Clearly, K and $\tilde{K}$ are monosplines of degree r and, in view of the properties of the Bernoulli polynomials of odd degree,

$$K(\tau_i)=\tilde{K}(\tau_i)=0, \qquad i=0, 1, ..., N.$$

Moreover, K has $N/2-1=n-s-1$ simple knots in $(0, 1)$, located at $x_k=2k/N$, $k=1, ..., n-s-1$, and $\tilde{K}$ has $N/2$ simple knots in $(0, 1)$, located at $\tilde{x}_k=(2k-1)/N$, $k=1, ..., n-s$.

The functions $g := K - K^G_{n,r}$ and $\tilde{g} := \tilde{K} - K^G_{n,r}$ are spline functions of degree $r-1$ with simple knots in $(0, 1)$ only, namely, $\{x_k\}_{k=1}^{n-s-1} \cup \{\xi^G_k\}_{k=1}^n$ for g and $\{\tilde{x}_k\}_{k=1}^{n-s} \cup \{\xi^G_k\}_{k=1}^n$ for $\tilde{g}$. We want to apply the Budan-Fourier Theorem to g and $\tilde{g}$. Therefore, we need the signs of the derivatives of g and $\tilde{g}$ at $y=0$ and $y=1$. In view of (3.5), there holds

$$g^{(j)}(y) = -\left(\frac{2}{N}\right)^{r-j} B_{r-j}(y), \qquad j = 0, ..., r-1, \tag{3.6}$$

$$\tilde{g}^{(j)}(y) = -\left(\frac{2}{N}\right)^{r-j} B_{r-j}\left(\frac{1}{2}\right), \qquad j = 0, ..., r-1 \tag{3.7}$$

with $y = 0, 1$. Now, using the sign properties (ii)–(v) from Lemma 2.1, we get from (3.6)–(3.7) (with $\sigma = 0$, $\tau = 1$ for g, but $\sigma > 0$, $\tau < 1$ for $\tilde{g}$)

$$\begin{aligned}
&S^-(g(0), ..., g^{(r-1)}(\sigma+)) \\
&\qquad = S^-(0, (-1)^s, 0, (-1)^{s-1}, ..., (-1)^1, (-1)^0) = s, \\
&S^+(g(1), ..., g^{(r-1)}(\tau-)) \\
&\qquad = S^+(0, (-1)^s, 0, (-1)^{s-1}, ..., (-1)^1, (-1)^1) = s, \\
&S^-(\tilde{g}(0), ..., \tilde{g}^{(r-1)}(\sigma+)) \\
&\qquad = S^-(0, (-1)^{s-1}, 0, (-1)^{s-2}, ..., (-1)^0, *) \leqslant s, \\
&S^+(\tilde{g}(1), ..., \tilde{g}^{(r-1)}(\tau-)) \\
&\qquad = S^+(0, (-1)^{s-1}, 0, (-1)^{s-2}, ..., (-1)^0, *) \geqslant s
\end{aligned}$$

(the entries marked by $*$ are unknown), and Theorem 2.1 (Budan-Fourier) implies

$$N-1 \leqslant Z_g(0, 1) \leqslant Z_{g^{(r-1)}}(0, 1) \qquad \text{and} \qquad N-1 \leqslant Z_{\tilde{g}}(0, 1) \leqslant Z_{\tilde{g}^{(r-1)}}(0, 1) \tag{3.8}$$

(the lower bound follows from the fact that K, $\tilde{K}$ and $K^G_{n,r}$ have common zeros at $\tau_1, ..., \tau_{N-1}$). However, it is not immediately clear that the Budan-Fourier Theorem can be applied to $\tilde{g}$, since $\tilde{g}^{(r-1)}$ vanishes identically near the end points, and therefore $\tilde{g}$ might not be a spline of extract degree $r-1$ (which would mean that $\tilde{g} \in \pi_{r-2}$). But for $x \in (0, 1)$, there holds

$$\tilde{g}^{(r-1)}(x) = \frac{2}{N} \sum_{k=1}^{n-s} (x - \tilde{x}_k)^0_+ - \sum_{k=1}^{n} a^G_k (x - \xi^G_k)^0_+. \tag{3.9}$$

We observe from (3.9) that $\tilde{g}^{(r-1)}$ has $n-s$ positive, but n negative jumps, so that they cannot cancel each other out. Thus, $\tilde{g}$ is a spline of exact degree $r-1$. Analogously, for $x \in (0, 1)$ we have the representation

$$g^{(r-1)}(x) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{n-s-1} (x - x_k)_+^0 - \sum_{k=1}^{n} a_k^G (x - \xi_k^G)_+^0. \tag{3.10}$$

The identities (3.9) and (3.10) show that $g^{(r-1)}$ and $\tilde{g}^{(r-1)}$ may change sign from -1 to $+1$ only at the points $\{x_k\}_{k=1}^{n-s-1}$ and $\{\tilde{x}_k\}_{k=1}^{n-s}$, respectively. This, coupled with $g^{(r-1)}(0+) > 0$, $g^{(r-1)}(1-) < 0$ and $\tilde{g}^{(r-1)}(0+) = 0$, $\tilde{g}^{(r-1)}(1-) = 0$, implies

$$Z_{g^{(r-1)}}(0, 1) \leqslant 2(n-s-1) + 1 = N - 1, \tag{3.11}$$

$$Z_{\tilde{g}^{(r-1)}}(0, 1) \leqslant 2(n-s) - 1 = N - 1. \tag{3.12}$$

Finally, the comparison of (3.11) and (3.12) with (3.8) yields

$$Z_g(0, 1) = Z_{\tilde{g}}(0, 1) = N - 1.$$

Thus, both g and $\tilde{g}$ vanish in $(0, 1)$ at the points $\{\tau_i\}_{i=1}^{N-1}$ only, and all these zeros are simple zeros. But a simple zero in the sense of [3] is also a sign change, so that $g, \tilde{g}, K, \tilde{K}$ and $K_{n,r}^G$ change exactly at the τ_i. Hence, from

$$\operatorname{sign} g(\varepsilon) = \operatorname{sign} K(\varepsilon) = \operatorname{sign} g'(0) = (-1)^s, \tag{3.13}$$

$$\operatorname{sign} \tilde{g}(\varepsilon) = \operatorname{sign} \tilde{K}(\varepsilon) = \operatorname{sign} \tilde{g}'(0) = (-1)^{s-1} \tag{3.14}$$

for sufficiently small $\varepsilon > 0$, we obtain

$$\operatorname{sign} K(x) = \operatorname{sign}(K(x) - K_{n,r}^G(x)) = (-1)^{s+i-1}, \tag{3.15}$$

$$\operatorname{sign} \tilde{K}(x) = \operatorname{sign}(\tilde{K}(x) - K_{n,r}^G(x)) = (-1)^{s+i-2} \tag{3.16}$$

for $x \in (\tau_{i-1}, \tau_i)$, $i = 1, ..., N$ (with $\tau_0 := 0$, $\tau_N := 1$). It remains to take into consideration that for odd r $K_{n,r}^G(\varepsilon) < 0$ for sufficiently small $\varepsilon > 0$ (see (2.4)), to conclude that for odd s K and $K_{n,r}^G$ have the same orientation, i.e.,

$$\operatorname{sign} K_{n,r}^G(x) = \operatorname{sign} K(x) \qquad \text{for} \quad x \in [0, 1],$$

and for even s

$$\operatorname{sign} K_{n,r}^G(x) = \operatorname{sign} \tilde{K}(x) \qquad \text{for} \quad x \in [0, 1].$$

Then (3.15) and (3.16) imply

$$|K^G_{n,r}(x)| \leqslant |K(x)| \qquad \text{for all} \quad x \in [0, 1], \quad \text{if } r = 4m-1,$$

$$|K^G_{n,r}(x)| \leqslant |\tilde{K}(x)| \qquad \text{for all} \quad x \in [0, 1], \quad \text{if } r = 4m+1,$$

with equality sign only for $x = \tau_i$, $i = 0, ..., N$. The last inequalities imply

$$\|K^G_{n,r}\|_q < \|K\|_q = \|\tilde{K}\|_q \qquad \text{for} \quad 1 \leqslant q \leqslant \infty,$$

and the calculation of $\|K\|_q$ proves the statement of Theorem 1.1 for odd r.

Case B: $r = 2s$. In this case $N = 2(n-s)+1$. The monosplines which will be compared with $K^G_{n,r}$ are L and $\tilde{L}$ as defined in (3.3) and (3.4). Both L and $\tilde{L}$ vanish at $\tau_i = i/N$ for $i = 1, ..., N-1$. Moreover, L and $\tilde{L}$ have only simple knots in $(0, 1)$, located at

$$y_i = \frac{4i-1}{2N}, \quad i = 1, ..., n-s \qquad \text{and} \qquad \tilde{y}_i = \frac{4i-3}{2N}, \quad i = 1, ..., n-s+1,$$

respectively. Let $h := L - K^G_{n,r}$ and $\tilde{h} := \tilde{L} - K^G_{n,r}$. Clearly h and $\tilde{h}$ are spline functions of degree $r-1$. Moreover, $h(0) = h(1) = \tilde{h}(0) = \tilde{h}(1) = 0$, and, in view of (3.5),

$$h^{(j)}(0) = \tilde{h}^{(j)}(1) = \left(\frac{2}{N}\right)^{r-j} B_{r-j}\left(\frac{1}{4}\right), \qquad j = 1, ..., r-1, \tag{3.17}$$

$$h^{(j)}(1) = \tilde{h}^{(j)}(0) = \left(\frac{2}{N}\right)^{r-j} B_{r-j}\left(\frac{3}{4}\right), \qquad j = 1, ..., r-1. \tag{3.18}$$

Then, taking into account Lemma 2.1 (v)–(vi), we obtain (with $\sigma = 0$ and $\tau = 1$)

$$\begin{aligned}
&S^-(h(0), ..., h^{(r-1)}(0+)) \\
&\qquad = S^-(0, (-1)^s, ..., (-1)^2, (-1)^1, (-1)^1) = s-1, \\
&S^+(h(1), ..., h^{(r-1)}(1-)) \\
&\qquad = S^+(0, (-1)^{s-1}, ..., (-1)^1, (-1)^1, (-1)^0) = s, \\
&S^-(\tilde{h}(0), ..., \tilde{h}^{(r-1)}(0+)) \\
&\qquad = S^-(0, (-1)^{s-1}, ..., (-1)^1, (-1)^1, (-1)^0) = s-1, \\
&S^+(\tilde{h}(1), ..., \tilde{h}^{(r-1)}(1-)) \\
&\qquad = S^+(0, (-1)^s, ..., (-1)^2, (-1)^1, (-1)^1) = s.
\end{aligned}$$

Now we apply the Budan-Fourier Theorem to obtain

$$N-1 \leqslant Z_f(0,1) \leqslant Z_{f^{(r-1)}}(0,1)-1 \qquad \text{for } f=h, \tilde{h}. \tag{3.19}$$

However, for $x \in (0,1)$

$$h^{(r-1)}(x) = -\frac{1}{2N} - \frac{2}{N}\sum_{k=1}^{n-s}(x-y_k)_+^0 + \sum_{k=1}^{n} a_k^G (x-\xi_k^G)_+^0,$$

$$\tilde{h}^{(r-1)}(x) = \frac{1}{2N} - \frac{2}{N}\sum_{k=1}^{n-s+1}(x-\tilde{y}_k)_+^0 + \sum_{k=1}^{n} a_k^G (x-\xi_k^G)_+^0,$$

and, as in Case A, we conclude that $h^{(r-1)}$ and $\tilde{h}^{(r-1)}$ may change sign from $+1$ to -1 only at the points $\{y_k\}_{k=1}^{n-s}$ and $\{\tilde{y}_k\}_{k=1}^{n-s+1}$, respectively. This, coupled with sign $h^{(r-1)}(\varepsilon) = \text{sign } \tilde{h}^{(r-1)}(1-\varepsilon) = -1$ and sign $h^{(r-1)}(1-\varepsilon) = \text{sign } \tilde{h}^{(r-1)}(\varepsilon) = 1$ for sufficiently small $\varepsilon > 0$ implies

$$Z_{f^{(r-1)}} \leqslant 2(n-s)+1 = N \qquad \text{for } f=h, \tilde{h},$$

and from (3.19) we infer that h and $\tilde{h}$ have no other zeros in $(0,1)$ except $\{\tau_i\}_{i=1}^{N-1}$, and the τ_i are simple zeros of h and $\tilde{h}$. As in Case A, we obtain

$$\text{sign } L(x) = \text{sign}(L(x) - K_{n,r}^G(x)) = (-1)^{s+i-1},$$

$$\text{sign } \tilde{L}(x) = \text{sign}(\tilde{L}(x) - K_{n,r}^G(x)) = (-1)^{s+i-2}$$

for $x \in (\tau_{i-1}, \tau_i)$ and $i = 1, ..., N$. For even r, $K_{n,r}^G(\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. Proceeding as in Case A, we finally obtain

$$|K_{n,r}^G(x)| \leqslant |L(x)| \quad \text{if } s \text{ is even}, \qquad |K_{n,r}^G(x)| \leqslant |\tilde{L}(x)| \quad \text{if } s \text{ is odd},$$

and

$$\|K_{n,r}^G\|_q < \|L\|_q \quad \text{if } r=4m, \qquad \|K_{n,r}^G\|_q < \|\tilde{L}\|_q \quad \text{if } r=4m+2.$$

A calculation of $\|L\|_q$ and $\|\tilde{L}\|_q$, using elementary properties of the Bernoulli polynomials, completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. The Peano kernel $K_{n+1,r}^L$ of the Lobatto quadrature formula (1.8) vanishes at $\{\tau_i\}_{i=1}^{N-1}$ and satisfies (3.5) for $j = 0, ..., r-2$. Instead of zero boundary conditions for the $(r-1)$st derivative we have

$$(K_{n+1,r}^L)^{(r-1)}(0) = (-1)^{r+1} a_0^L, \; (K_{n+1,r}^L)^{(r-1)}(1) = (-1)^r a_0^L$$

($a_0^L = a_n^L$, by symmetry). For convenience's sake we use the notation from the proof of Theorem 1.1.

Case A: $r=2s+1$. We compare $K^L_{1+n,r}$ with the monosplines (3.1)–(3.2), and set $g:=K-K^L_{n+1,r}$, $\tilde{g}:=\tilde{K}-K^L_{n+1,r}$. Then (3.6)–(3.7) remain valid for $j=0, ..., r-2$, but for the $(r-1)$st derivatives we have

$$g^{(r-1)}(0)=-g^{(r-1)}(1)=\frac{1}{N}-a^L_0 \quad \text{and} \quad \tilde{g}^{(r-1)}(0)=-\tilde{g}^{(r-1)}(1)=-a^L_0,$$

which yields

$$S^-(g(0), ..., g^{(r-1)}(\sigma+))\leqslant s,$$
$$S^+(g(1), ..., g^{(r-1)}(\tau-))\geqslant s,$$
$$S^-(\tilde{g}(0), ..., \tilde{g}^{(r-1)}(0+))=s,$$
$$S^+(\tilde{g}(1), ..., \tilde{g}^{(r-1)}(1-))=s.$$

Again, (3.8) holds. Since, for $x\in(0, 1)$,

$$g^{(r-1)}(x)=\frac{1}{N}-a^L_0+\frac{2}{N}\sum_{k=1}^{n-s-1}(x-x_k)^0_+-\sum_{k=1}^{n-1}a^L_k(x-\xi^L_k)^0_+,$$

we obtain for $a^L_0\geqslant 1/N$ that

$$Z_{g^{(r-1)}}(0, 1)\leqslant 2(n-s-1)-1=N-3,$$

and for $a^L_0<1/N$ that

$$Z_{g^{(r-1)}}(0, 1)\leqslant 2(n-s-1)+1=N-1.$$

The first case is impossible, since $Z_g(0, 1)\geqslant N-1$, so that

$$a^L_0<\frac{1}{N} \tag{3.20}$$

and $Z_g(0, 1)=N-1$ must hold. The same is obtained for $\tilde{g}$. Then, proceeding in the same way as in the proof of Theorem 1.1, we deduce that both g and $\tilde{g}$ vanish in $(0, 1)$ at $\{\tau_i\}_{i=1}^{N-1}$ only, all these zeros are simple and therefore for all $x\in[0, 1]$

$$|K^L_{n+1,r}(x)|\leqslant|K(x)| \qquad \text{if} \quad r=4m+1,$$
$$|K^L_{n+1,r}(x)|\leqslant|\tilde{K}(x)| \qquad \text{if} \quad r=4m-1,$$

with equality sign only for $x=\tau_i$, $i=0, ..., N$.

Case B: $r=2s$. We compare $K^L_{n+1,r}$ with the monosplines L and $\tilde{L}$, defined by (3.3) and (3.4). Define $h:=L-Q^L_{n+1,r}$ and $\tilde{h}:=\tilde{L}-Q^L_{n+1,r}$, then h and $\tilde{h}$ are splines of degree $r-1$ with simple knots in $(0, 1)$ $\{y_i\}_{i=1}^{n-s} \cup \{\xi_i^L\}_{i=1}^{n-1}$ and $\{\tilde{y}_i\}_{i=1}^{n-s} \cup \{\xi_i^L\}_{i=1}^{n-1}$, respectively. Both h and $\tilde{h}$ vanish at $\{\tau_i\}_{i=0}^{N}$, and satisfy (3.17) and (3.18) for $j=1, ..., r-2$. For $j=r-1$ we have

$$h^{(r-1)}(0) = -h^{(r-1)}(1) = -\frac{1}{2N} + a_0^L,$$

$$\tilde{h}^{(r-1)}(0) = -h^{(r-1)}(1) = \frac{1}{2N} + a_0^L.$$

Further, the application of Lemma 2.1(iv)–(vi) yields

$$S^-(h(0), ..., h^{(r-1)}(\sigma+)) = s-1+\delta, \tag{3.21}$$

$$S^+(h(1), ..., h^{(r-1)}(\tau-)) = s-\delta, \tag{3.22}$$

$$S^-(\tilde{h}(0), ..., \tilde{h}^{(r-1)}(0+)) = s-1, \tag{3.23}$$

$$S^+(\tilde{h}(1), ..., \tilde{h}^{(r-1)}(1-)) = s, \tag{3.24}$$

where $\delta=1$ if $a_0^L > 1/(2N)$, $\delta=0$ if $a_0^L < 1/(2N)$, and $\delta \in \{0, 1\}$ if $a_0^L = 1/(2N)$ (i.e., the exact value of δ is not known in this case). Using the representation

$$h^{(r-1)}(x) = a_0^L - \frac{1}{2N} - \frac{2}{N}\sum_{k=1}^{n-s} (x-y_k)_+^0 + \sum_{k=1}^{n-1} a_k^L(x-\xi_k^L)_+^0, \tag{3.25}$$

we examine separately the cases $a_0^L \geqslant 1/(2N)$ and $a_0^L < 1/(2N)$. If $a_0^L \geqslant 1/(2N)$, we conclude by (3.25) that

$$Z_{h^{(r-1)}}(0, 1) \leqslant 2(n-s)-1 = N-2,$$

and then (3.21)–(3.22) and the Budan-Fourier Theorem yield

$$N-1 \leqslant Z_h(0, 1) \leqslant N-1. \tag{3.26}$$

If $a_0^L < 1/(2N)$, then

$$Z_{h^{(r-1)}}(0, 1) \leqslant 2(n-s)+1 = N,$$

and again from the Budan-Fourier Theorem and (3.21)–(3.22) we arrive at (3.26). The verification that h is a spline of exact degree $r-1$ is straightforward for $s>1$, while in the case $s=1$, it turns out that $h \equiv 0$, i.e., $K_{n+1,2}^L \equiv L$ (see [12], Corollary 1).

Thus, in all cases we obtain that h can not have other zeros in $(0, 1)$ except $\{\tau_i\}_{i=1}^{N-1}$, and these zeros are simple. Using the same reasoning, we establish this for $\tilde{h}$, too. The proof of Case **B** is completed by the same arguments as before. ▮

Proof of Theorem 1.3. The proof of Theorem 1.3 is analogous to those of Theorems 1.1 and 1.2. With $N = 2n + 2 - r$, we compare $K_{n+1,r}^{R0}$ with the monosplines (3.1)–(3.2) in the case of odd r, and with the monosplines (3.3)–(3.4) in the other case. The error bound for $R_{n+1,r}^{R0}$ is the same as for $R_{n+1,r}^{R1}$, by arguments of symmetry. We omit the details. ▮

Proof of Theorem 1.4. For odd r and arbitrary p, the theorem follows immediately from a comparison of the estimates of Theorems 1.1–1.3 with (1.3).

For even r and $p = \infty$, $q = 1$, it follows in the same way, because, in this case, the infimum in (1.2) is attained for $c = B_r(1/4)$. ▮

Proof of Theorem 1.5. In the proofs of Theorems 1.1–1.3 we established that the Budan-Fourier Theorem holds with an equality sign for the functions g, $\tilde{g}$, h and $\tilde{h}$ considered there. This forces their derivatives $g^{(r-1)}$, $\tilde{g}^{(r-1)}$, $h^{(r-1)}$ and $\tilde{h}^{(r-1)}$ to have the maximal possible number of sign changes in $(0, 1)$, which will be used to establish the inequalities (1.11)–(1.14). We illustrate this for the Lobatto case only, the remaining cases are proved analogously.

In the case $r = 2s + 1$, we have

$$g^{(r-1)}(x) = \frac{1}{N} - a_0^L + \frac{2}{N} \sum_{k=1}^{n-s-1} \left(x - \frac{2k}{N}\right)_+^0 - \sum_{k=1}^{n-1} a_k^L (x - \xi_k^L)_+^0, \qquad (3.27)$$

$$\tilde{g}^{(r-1)}(x) = -a_0^L + \frac{2}{N} \sum_{k=1}^{n-s} \left(x - \frac{2k-1}{N}\right)_+^0 - \sum_{k=1}^{n-1} a_k^L (x - \xi_k^L)_+^0, \qquad (3.28)$$

and, following the proof of Theorem 1.2, $g^{(r-1)}$ must have sign changes from $+$ to $-$ at $2k/N$, $k = 1, ..., n-s-1$, and $\tilde{g}^{(r-1)}$ at $(2k-1)/N$, $k = 1, ..., n-s$. Actually, the inequality (1.12) was already proved for $l = 0$ (see (3.20)), and it follows from (3.27) that there must hold

$$\frac{1}{N} + \frac{2(k-1)}{N} - \sum_{0 \leqslant \xi_\nu^L < 2k/N} a_\nu^L < 0 \qquad \text{for } k = 1, ..., n-s-1, \qquad (3.29)$$

$$\frac{1}{N} + \frac{2l}{N} - \sum_{0 \leqslant \xi_\nu^L \leqslant 2l/N} a_\nu^L > 0 \qquad \text{for } l = 1, ..., n-s-1. \qquad (3.30)$$

Similarly, from (3.28) we get

$$\frac{2(k-1)}{N} - \sum_{0 \leq \xi_\nu^L < (2k-1)/N} a_\nu^L < 0 \quad \text{for} \quad k = 1, ..., n-s, \tag{3.31}$$

$$\frac{2l}{N} - \sum_{0 \leq \xi_\nu^L \leq (2l-1)/N} a_\nu^L > 0 \quad \text{for} \quad l = 1, ..., n-s. \tag{3.32}$$

It is easily seen that (3.29) and (3.31) are exactly the inequalities (1.11), and (3.30) and (3.32) are exactly the inequalities (1.12) for the Lobatto case.

Analogously, for $r = 2s \geq 4$ (for $r = 2$, the following holds with equality sign in (3.33) and (3.34)) we consider

$$h^{(r-1)}(x) = a_0^L - \frac{1}{2N} - \frac{2}{N} \sum_{k=1}^{n-s} \left(x - \frac{4k-1}{2N}\right)_+^0 + \sum_{k=1}^{n-1} a_k^L (x - \xi_k^L)_+^0,$$

$$\tilde{h}^{(r-1)}(x) = a_0^L + \frac{1}{2N} - \frac{2}{N} \sum_{k=1}^{n-s+1} \left(x - \frac{4k-3}{2N}\right)_+^0 + \sum_{k=1}^{n-1} a_k^L (x - \xi_k^L)_+^0.$$

Since $h^{(r-1)}$ and $\tilde{h}^{(r-1)}$ must change sign at $(4k-1)/(2N)$ and $(4k-3)/(2N)$, respectively, there must hold

$$-\frac{1}{2N} - \frac{2(k-1)}{N} + \sum_{0 \leq \xi_\nu^L < (4k-1)/(2N)} a_\nu^L > 0 \quad \text{for} \quad k = 1, ..., n-s, \tag{3.33}$$

$$-\frac{1}{2N} - \frac{2l}{N} + \sum_{0 \leq \xi_\nu^L \leq (4l-1)/(2N)} a_\nu^L < 0 \quad \text{for} \quad l = 1, ..., n-s, \tag{3.34}$$

$$-\frac{1}{2N} + \frac{2(k-1)}{N} - \sum_{0 \leq \xi_\nu^L < (4k-3)/(2N)} a_\nu^L < 0 \quad \text{for} \quad k = 1, ..., n-s+1, \tag{3.35}$$

$$-\frac{1}{2N} + \frac{2l}{N} - \sum_{0 \leq \xi_\nu^L \leq (4l-3)/(2N)} a_\nu^L > 0 \quad \text{for} \quad l = 1, ..., n-s+1. \tag{3.36}$$

Clearly (3.33) and (3.35) are exactly the inequalities (1.13), and (3.34) and (3.36) are the inequalities (1.14) for the Lobatto case. The proof of the remaining cases is similar. Theorem 1.5 is proved. ∎

4. Concluding Remarks

1. It is well known (see, e.g., [17]) that the Peano kernels of the optimal quadrature formulae in W_p^r (i.e., the monosplines of least L_q-deviation; cf. (2.5)) have maximal number of simple zeros in (0, 1), so that the

optimal quadrature formulae are also Gaussian formulae for spaces of splines with simple knots; however, since the optimal formulae are not known, except in some special cases, it is also not known for which spline spaces they are exact.

2. Theorem 1.4 does not cover all cases in which the Gauss type formulae are asymptotically optimal. This follows from the following theorem, which is a special case of the Theorems 1 and 2 in [9].

THEOREM 4.1. *Let Q be a quadrature formula of the type* (1.1), *with non-negative weights and algebraic degree of precision at least $r-1$. Then*

$$\mathscr{E}(Q, W^j_\infty) \leqslant K_j p^{1/(j+1)} \left(\frac{1}{K_r} \mathscr{E}(Q, W^r_\infty)\right)^{j/r}$$

for $1 \leqslant j \leqslant r-1$, where $p = 1$ for odd r, $p = 2$ for even r, and K_j are Favard's constants (1.5).

In view of Theorem 1.4 and (1.4), this yields the following corollary.

COROLLARY 4.1. *The Gauss type formulae related to the spaces $S_{r-1,N}$ are also asymptotically optimal in W^j_∞ for odd j with $1 \leqslant j \leqslant r-1$.*

3. By taking the differences of (1.11) and (1.12), and of (1.13) and (1.14), we obtain the following corollary.

COROLLARY 4.2. *With * standing for G, L and $R0$, there holds*

$$\sum_{l/N < \xi_v^* < k/N} a_v^* > \frac{k-l-2}{N} \qquad \text{for} \quad 0 \leqslant l < k \leqslant N \text{ and odd } r \geqslant 3,$$

and

$$\sum_{(2l-1)/(2N) < \xi_v^* < (2k-1)/(2N)} a_v^* > \frac{k-l-2}{N} \qquad \text{for} \quad 1 \leqslant l < k \leqslant N \text{ and even } r \geqslant 2.$$

For similar estimates for Gauss type quadrature formulae related to spaces of polynomials (the standard case), see Förster [6].

A a consequence of Corollary 4.2 and Theorem 1.5, for odd $r \geqslant 3$ each of the intervals $((k-2)/N, k/N) \cap [0, 1]$, $k = 1, ..., N+1$ must contain at least one node of the Gauss type formulae related to $S_{r-1,N}$, and the same is true for even $r \geqslant 2$ and the intervals $((2k-5)/(2N), (2k-1)/(2N)) \cap [0, 1]$, $k = 2, ..., N+1$.

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